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Journal of Geometry and Physics 21 (1997) 218–230

JOURNAL OF GEOMETRY AND PHYSICS

# Hyperkähler metrics of cohomogeneity one

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Received 8 February 1996

**Abstract**

Irreducible hyperkähler manifolds of dimension greater than four admitting a cohomogeneity-one action of a compact simple Lie group are classified via coadjoint orbits. It is shown that the only complete example is the Calabi metric on $T^*\mathbb{C}P(n)$.

*Subj. Class.:* Differential geometry; General relativity

*1991 MSC:* 53C25, 57S25, 22E60

*Keywords:* Einstein equations; Hyperkähler metrics; Cohomogeneity; Calabi metric

## 1. Introduction

The intractability of the Einstein equations has led mathematicians interested in Einstein metrics to impose simplifying assumptions. One such assumption is that of homogeneity, when the Einstein condition may be expressed purely algebraically. Another natural restriction is to consider metrics of cohomogeneity one, that is, where the generic orbit of the isometry group has real codimension one. For cohomogeneity-one metrics the Einstein condition becomes a system of non-linear ordinary differential equations. Although considerable progress has been made [5,13], a classification of complete cohomogeneity-one Einstein metrics still seems far off.

In this paper we shall prove a classification theorem for cohomogeneity-one metrics which satisfy the stronger condition of being *hyperkähler*. This condition means that the metric is simultaneously Kähler with respect to three complex structures satisfying the quaternionic multiplication relations. Hyperkähler manifolds have dimension $4n$, their holonomy lies

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*PII* S0393-0440(96)00017-4

in $Sp(n)$ and they are Ricci-flat. The last property implies that homogeneous hyperkähler metrics are flat, so in some sense the cohomogeneity one examples should be the simplest non-trivial hyperkähler metrics.

We shall prove the following theorem.

**Theorem 1.1.** *Let $M$ be an irreducible hyperkähler manifold of dimension greater than four, of cohomogeneity one with respect to a compact simple Lie group $G$. Then $M$ is an open subset of*

(i) *the cotangent bundle of complex projective space with the Calabi metric* [10], *or*

(ii) *the space $\mathcal{U}(N)$ of Swann* [18] *(with its standard metric), where $N$ is a compact Wolf space.*

*If $M$ is complete, then $M$ is $T^* \mathbb{C}\mathrm{P}(n)$ with the Calabi metric.*

**Remark 1.2.** Our assumptions are chosen to force the group action to preserve each complex structure. This is because irreducibility implies that the space of parallel two-forms has real dimension precisely equal to three. Our assumptions on $G$ and $\dim M$ mean that this space is acted on trivially by $G$. In other words, $G$ fixes each complex structure.

If $M$ is four-dimensional and $G$ is a compact simple group, then all the cohomogeneity-one hyperkähler metrics are known [4,3,14]. There are two families, both of cohomogeneity one with respect to $SU(2)$. In one family the complex structures are all fixed by the action, while in the other, $SU(2)$ acts on the space of parallel two-forms via the adjoint representation, so acts transitively on the two-sphere of complex structures. The only complete example in the first family (except for Euclidean space) is the Eguchi–Hanson metric on $T^* \mathbb{C}\mathrm{P}(1)$, which is included in the examples of Calabi. However the first family also includes a two-parameter set of incomplete examples, classified in [4], which are not restrictions of the Eguchi–Hanson metric. This contrasts with the situation in dimension greater than four described by our theorem.

**Remark 1.3.** Our proof of Theorem 1.1 also goes through in the case when $G$ is compact and semi-simple, provided one assumes that any $\mathfrak{su}(2)$-factor in the decomposition of $\mathfrak{g}$ into simple algebras acts trivially on the complex structures. One may also apply the result to actions of general compact groups if one makes the above assumption not only for $\mathfrak{su}(2)$-factors but also for Abelian factors, and if in addition one assumes the existence of a hyperkähler moment map.

**Remark 1.4.** Bielawski [7] has informed us of an alternative proof under the additional assumption that the hyperkähler metrics are complete.

## 2. Complex structures

From now on we shall assume that $M$ and $G$ are as in the statement of Theorem 1.1.

There will be an open dense set $\widehat{M}$ in $M$ consisting of the union of the principal orbits. These orbits are copies of $G/K$ for some closed subgroup $K$ of $G$, so our open set may be identified topologically with $I \times G/K$ for an open subinterval $I$ of the real line. Note that by the Cheeger–Gromoll theorem [11], if our metric is to be complete there must also be precisely one non-principal orbit $G/H$. (If there were two non-principal orbits, then $M$ would be compact so the Killing fields would be parallel and the isometry group could not be simple.) Since $M$ is a manifold, $H/K$ is necessarily a sphere.

Let $J$ be a complex structure, and let $G_{\mathbb{C}}$ denote the complexification of $G$ with respect to $J$. The space of $J$-complexified Killing fields spans the tangent space at each point of $\widehat{M}$, so we may equivariantly identify an open $G$-invariant neighbourhood of a principal orbit in $\widehat{M}$ with a $G$-invariant open set in some $G_{\mathbb{C}}$-homogeneous space $Q$. Moreover, as $G$ acts $J$-holomorphically and preserves the complex-symplectic forms on $M$, we see that $Q$ is homogeneous complex-symplectic with respect to $G_{\mathbb{C}}$. Since $G$ is simple, the hyperkähler moment map for the $G$-action on $M$ exists and hence the complex-symplectic moment map for the $G_{\mathbb{C}}$-action on $Q$ exists also.

A theorem of Lichnerowicz [17] now tells us that $Q$ is a cover of a coadjoint orbit $\mathcal{O}$ of $G_{\mathbb{C}}$, which because of our hypotheses on $M$ must be of cohomogeneity one with respect to $G$. We must now analyse when this situation can occur.

**Theorem 2.1.** *Let $\mathcal{O}$ be a coadjoint orbit of $G_{\mathbb{C}}$, where $G$ is a compact simple group. If $\mathcal{O}$ is of cohomogeneity one with respect to $G$, then it is either the nilpotent orbit $\mathcal{U}(N)$, where $N$ is the Wolf space associated to $G$, or $SL(n, \mathbb{C})/GL(n-1, \mathbb{C})$.*

*Proof.* Let $\mathfrak{g}_{\mathbb{C}}$ be the Lie algebra of $G_{\mathbb{C}}$ and identify $\mathfrak{g}_{\mathbb{C}}$ with $\mathfrak{g}_{\mathbb{C}}^*$ via the Killing form. Under this identification the coadjoint orbits correspond to the adjoint orbits. Let $X$ be an element of $\mathfrak{g}_{\mathbb{C}}$. The Jordan–Chevally decomposition (see [15]) allows us to write $X = X_{\mathrm{s}} + X_{\mathrm{n}}$ in a unique way with $X_{\mathrm{s}}$ semi-simple, $X_{\mathrm{n}}$ nilpotent and $[X_{\mathrm{s}}, X_{\mathrm{n}}] = 0$. Uniqueness implies that $\mathrm{stab}_{G_{\mathbb{C}}} X = \mathrm{stab}_{G_{\mathbb{C}}} X_{\mathrm{s}} \cap \mathrm{stab}_{G_{\mathbb{C}}} X_{\mathrm{n}}$.

Let us write $\mathrm{cohom}\, X$ for the cohomogeneity of the $G_{\mathbb{C}}$-orbit $\mathcal{O}$ of $X$ under the action of $G$. Then we claim

$$\mathrm{cohom}\, X \geq \max\{\mathrm{cohom}\, X_{\mathrm{s}}, \mathrm{cohom}\, X_{\mathrm{n}}\}. \tag{2.1}$$

To prove this, suppose $\varphi : N_1 \to N_2$ is a smooth, equivariant surjection between two $G$-manifolds. For $i = 1, 2$, let $U_i \subset N_i$ be the open dense set which is the union of principal orbits. Then $\varphi^{-1}(U_2)$ is open in $N_1$ and $G$-invariant, so it is enough to consider $\varphi$ restricted to $V_1 := U_1 \cap \varphi^{-1}(U_2)$. Let $V_2 = \varphi(V_1)$. Then $V_i/G$, $i = 1, 2$, are manifolds whose dimension is the cohomogeneity of the $G$-action on $N_i$. But the map $V_1/G \to V_2/G$ induced by $\varphi$ is a smooth surjection, so it decreases dimension. Hence, we have $\mathrm{codim}\, N_1 \geq \mathrm{codim}\, N_2$. Inequality (2.1) is now seen to hold by applying the above arguments to the equivariant surjections $\mathcal{O} \to \mathcal{O}_{\mathrm{s}} := G_{\mathbb{C}} \cdot X_{\mathrm{s}}$ and $\mathcal{O} \to \mathcal{O}_{\mathrm{n}} := G_{\mathbb{C}} \cdot X_{\mathrm{n}}$ given by taking the semi-simple and nilpotent parts.

Thus, by (2.1), to find the orbits of cohomogeneity one, we first have to identify the nilpotent orbits $\mathcal{O}_{\mathrm{n}}$ and semi-simple orbits $\mathcal{O}_{\mathrm{s}}$ which have cohomogeneity at most one.

If $\mathcal{O}_{\mathrm{n}}$ is a non-zero nilpotent orbit then it is non-compact and so cannot be homogeneous under the action of $G$. In [18], it was shown that $\mathcal{O}_{\mathrm{n}}$ admits a $G$-invariant hyperkähler metric and an action of $\mathbb{H}^*$ such that $N = \mathcal{O}_n/\mathbb{H}^*$ is a quaternionic Kähler manifold and $G$ acts isometrically on $N$. If $\mathcal{O}_n$ is of cohomogeneity one, then $N$ is homogeneous and it follows from [1,2] that $N$ is a compact symmetric quaternionic Kähler manifold, i.e. a Wolf space [19], and that $\mathcal{O}$ is the (unique) minimal nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$.

Now consider the semi-simple orbit $\mathcal{O}_s$. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_{\mathbb{C}}$ containing $X_s$, let $\Delta$ be the set of roots and let $\Delta^+$ be the set of positive roots. Using the action of the Weyl group we may assume that $X_s$ lies in the positive Weyl chamber. Write $\mathfrak{g}_\alpha$ for the root space associated to $\alpha$ and let $\sigma$ be the real structure such that $\mathfrak{g} = (\mathfrak{g}_{\mathbb{C}})^\sigma$ and $\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$. Then

$$\operatorname{stab}_{\mathfrak{g}_{\mathbb{C}}} X_s = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha\in\Delta\\ \alpha(X_s)=0}} \mathfrak{g}_\alpha,$$

which is clearly $\sigma$-invariant. Thus the real orbit through $X_s$ has half the (real) dimension of the complex orbit. However, this is not a principal orbit for the action of $G$ on $\mathcal{O}_s$.

The $G$-action is of cohomogeneity one if and only if the action of $\operatorname{stab}_G X_s$ on non-zero vectors at $X_s$ normal to $G \cdot X_s$ in $\mathcal{O}_s$ is also of cohomogeneity one. The tangent space to $\mathcal{O}_s$ at $X_s$ is $\operatorname{ad}(X_s)\mathfrak{g}_{\mathbb{C}}$. Let $\alpha$ be a simple root such that $\alpha(X_s)$ is non-zero and let $E_\alpha \in \mathfrak{g}_\alpha$ be the corresponding element of the Cartan basis (or indeed any non-zero element in $\mathfrak{g}_\alpha$). Then $[X_s, E_\alpha] = \alpha(X_s)E_\alpha$, so $E_\alpha$ lies in the tangent space to $\mathcal{O}_s$ at $X_s$. Note that $X_s$ is in the positive Weyl chamber, so $\alpha(X_s) > 0$. Furthermore, $E_\alpha$ is not tangent to the real orbit through $X_s$ as it is not a linear combination of elements of the form $E_\alpha - E_{-\alpha}$. Write $Y \in \mathcal{O}$ for the image of this tangent vector under the exponential map. We have

$$\operatorname{stab}_{\mathfrak{g}_{\mathbb{C}}} E_\alpha = \{\, H \in \mathfrak{h}: \alpha(H) = 0 \,\} \oplus \bigoplus_{\substack{\beta\in\Delta\setminus\{-\alpha\}\\ \alpha+\beta\notin\Delta}} \mathfrak{g}_\beta \tag{2.2}$$

and the real part of this is the stabiliser for the $G$-action. On the other hand

$$\begin{aligned}\operatorname{codim}_{\mathcal{O}} G \cdot Y &= \dim G_{\mathbb{C}} - \dim \operatorname{stab}_{G_{\mathbb{C}}} X_s\\ &\quad - \dim G + \dim(\operatorname{stab}_G X_s \cap \operatorname{stab}_G E_\alpha)\\ &= \dim G - \dim \operatorname{stab}_G X_s - \dim(\operatorname{stab}_{\mathfrak{g}} X_s \ominus \operatorname{stab}_{\mathfrak{g}} E_\alpha),\end{aligned}$$

where $\ominus$ denotes the ortho-complement. Now

$$\dim G - \dim \operatorname{stab}_G X_s = 2\#\{\, \beta \in \Delta^+: \beta(X_s) \neq 0 \,\}$$

and from (2.2)

$$\begin{aligned}&\dim(\operatorname{stab}_{\mathfrak{g}} X_s \ominus \operatorname{stab}_{\mathfrak{g}} E_\alpha)\\ &\quad = 1 + 2\#\{\, \beta \in \Delta^+: \beta(X_s) = 0,\ \{\alpha \pm \beta\} \cap (\Delta \cup \{0\}) \neq \emptyset \}.\end{aligned}$$

However, $\beta(X_s) = 0$ implies $\alpha \pm \beta \neq 0$, so either $\alpha + \beta \in \Delta^+$ or $\alpha - \beta \in \Delta^-$. But $\alpha - \beta$ is positive on $X_s$ so cannot lie in $\Delta^-$. The codimension of the $G$-orbit through $Y$ is thus

$$\begin{aligned}
\operatorname{codim}_{\mathcal{O}} G \cdot Y &= 1 + 2\#\{\beta \in \Delta^+ \setminus \{\alpha\} : \beta(X_s) \neq 0\} \\
&\quad -2\#\{\beta \in \Delta^+ : \beta(X_s) = 0, \alpha + \beta \in \Delta^+\} \\
&= 1 + 2\#\{\beta \in \Delta^+ \setminus \{\alpha\} : \beta(X_s) \neq 0\} \\
&\quad -2\#\{\gamma \in \Delta^+ : \gamma(X_s) = \alpha(X_s), \gamma - \alpha \in \Delta^+\} \\
&= 1 + 2\#\left\{\begin{aligned}&\beta \in \Delta^+ \setminus \{\alpha\} : \beta(X_s) \neq 0, \\ &\qquad (\beta(X_s) = \alpha(X_s) \Longrightarrow \beta - \alpha \notin \Delta)\end{aligned}\right\}.
\end{aligned}$$

Thus $\mathcal{O}_s$ is of cohomogeneity one if and only if the latter set is empty, i.e. *for any positive root $\beta \neq \alpha$ either $\beta(X_s) = 0$ or $\beta(X_s) = \alpha(X_s)$ and $\beta - \alpha$ is a root*. This implies that $\alpha$ is the only simple root which is non-zero on $X_s$ and $\alpha$ occurs with multiplicity at most one in the expression of any positive root as a sum of simple roots. In particular, if there are short and long roots, then $\alpha$ is long. Moreover, $\alpha$ is an extreme root in the Dynkin diagram; since if $\alpha$ has two neighbours $\alpha'$ and $\alpha''$, then $\beta = \alpha + \alpha' + \alpha''$ is a root with $\beta(X_s) = \alpha(X_s)$, but $\beta - \alpha = \alpha' + \alpha''$ is not a root.

We now consider the different simple Lie algebras in turn.

$G_2$, $F_4$ and $E_8$ are ruled out because every simple root has multiplicity strictly greater than one in the top root forms $2 \Rrightarrow 3$, $23 \Rightarrow 42$ and $2345\overset{3}{6}42$, respectively.

In $B_n$, $C_n$, $D_n$, $E_6$ and $E_7$, the top root forms are $\beta = 12 \cdots 2 \Rightarrow 2$, $1 \Rightarrow 2 \cdots 2$, $12 \cdots 2^{1}_{1}$, $12\overset{2}{3}21$ and $123\overset{2}{4}32$, respectively. These all contain a long extreme simple root $\alpha$ of multiplicity one, but in each case $\beta - \alpha$ is not a root.

Thus we are left with type $A_n$, where there is only one extreme root $\alpha = 10\ldots0$, up to the action of the Weyl group, and we have $X_s = \operatorname{diag}(-n\lambda, \lambda, \ldots, \lambda)$. This orbit is $SL(n, \mathbb{C})/GL(n-1, \mathbb{C})$, as required.

Thus the only semi-simple orbit of cohomogeneity one occurs for type $A_n$. It is now easy to show that the combination of this orbit with the minimal nilpotent orbit has cohomogeneity strictly greater than one and the proof is complete. □

**Remark 2.2.** One may give a geometric argument for which semi-simple orbits are of cohomogeneity one as follows. The orbit $\mathcal{O}_s$ may be written as $G_{\mathbb{C}}/K_{\mathbb{C}}$ for some closed subgroup $K$ of $G$. This quotient is $G$-equivariantly diffeomorphic to the tangent bundle $T(G/K)$. If $\mathcal{O}_s$ is of cohomogeneity one, then $G$ acts transitively on the unit sphere bundle of $T(G/K)$. This implies that $G/K$ is a two-point homogeneous space and hence is a rank-one symmetric space [6, Section 7.15]. Now Biquard [8] and Kovalev [16] show that $\mathcal{O}_s$ admits a $G$-invariant hyperkähler metric such that the zero section $G/K$ is a complex submanifold of $T(G/K)$. Thus $G/K$ also admits a $G$-invariant Kähler metric. We now deduce that $G/K = \mathbb{C}P(n)$. This space has two homogeneous descriptions, namely

$$\frac{SU(n+1)}{U(n)} \quad \text{and} \quad \frac{Sp((n+1)/2)}{Sp((n-1)/2)U(1)},$$

but the latter does not occur as a semi-simple orbit.

## 3. Local calculations

We must analyse the hyperkähler metrics which can live on the complex manifolds described in Section 2. In this section we perform some preliminary calculations toward this end.

As mentioned earlier, the union of the principal orbits of $M$ forms an open dense subset $\widehat{M}$ of $M$, topologically equivalent to $I \times G/K$ where $I$ is an open interval. We may take $I$ to be a geodesic orthogonal to the orbits of $G$. The metric is then cast in the form

$$g = \mathrm{d}t^2 + g_t,$$

where $t$ is the arc-length coordinate on $I$, and $g_t$ is a $G$-homogeneous metric on $G/K$ for each $t$.

The tangent space to the orbit at $\{t\} \times eK$ may be identified with an Ad $K$-invariant complement $\mathfrak{p}$ to $\mathfrak{k}$ in $\mathfrak{g}$. We can assume that $G$ acts effectively on $G/K$, so that the adjoint action of $K$ on $\mathfrak{p}$ is effective. From now on, all calculations will be performed on $I \times (eK)$.

We decompose $\mathfrak{p}$ into a sum of $K$-modules:

$$\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_s,$$

where $\mathfrak{p}_0$ is acted on trivially by $K$, and the other summands are irreducible. The metric on $M$ is determined by a family of $K$-invariant inner products on $\mathfrak{p}$, depending on $t$.

Let $J_1$, $J_2$, $J_3$ be an anti-commuting triple of complex structures. We define vector fields $\xi_a$ $(a = 1, 2, 3)$ on $\widehat{M}$ by

$$\xi_a = J_a(\partial/\partial t). \tag{3.1}$$

Taking the covariant derivative of (3.1) shows that

$$J_a \nabla_{Y_i} \frac{\partial}{\partial t} = \nabla_{Y_i} \xi_a \tag{3.2}$$

for all tangent vectors $Y_i$.

As each complex structure $J_a$, $a = 1, 2, 3$, is $G$-invariant, we see that on $I \times (eK)$ each of the three mutually orthogonal vectors $\xi_a$ lies in $\mathfrak{p}_0$, so $\mathfrak{p}_0$ has dimension at least three. In fact, the analysis of the possible underlying manifolds for $M$ in Section 2 shows that $\mathfrak{p}_0$ is precisely three-dimensional in all cases. Moreover it is a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{su}(2)$.

The four-dimensional vector space spanned by $\mathfrak{p}_0$ and the normal vector $\partial/\partial t$ is a quaternionic subspace of the tangent space to $M$, closed under the Lie bracket. It follows that the family of inner products on $\mathfrak{p}_0 \cong \mathfrak{su}(2)$ actually defines an $SU(2)$-invariant hyperkähler metric on $I \times SU(2)$. Moreover the complex structures of this metric are each $SU(2)$-invariant. Such metrics have been completely analysed in [4]. Ricci-flatness enables us in the usual way (see [9], for example) to diagonalise the metric on $\mathfrak{p}_0$ for all $t$, with respect to a basis $\{ X_1, X_2, X_3 \}$ satisfying the relations $[X_1, X_2] = X_3$, etc. The coefficients $h_1^2$, $h_2^2$, $h_3^2$ of the diagonalised metric satisfy the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}(h_1 h_2) + h_3 = 0, \quad \text{and cyclically.} \tag{3.3}$$

After changes of variables these are equivalent to the familiar spinning top equations of [4].

The methods of [12] may be used to show that, for a suitable choice of $J_1$, $J_2$, $J_3$, we can take

$$\xi_a = \frac{1}{h_a} X_a. \tag{3.4}$$

on $I \times (eK)$.

Suppose that there exists a basis $Y_i$ of Killing fields for $\mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_s$, orthonormal with respect to the Killing form, which diagonalises the metric for all $t$, so that

$$g(Y_i, Y_j) = \delta_{ij} f_i^2$$

for functions $f_i$ of $t$. (A priori such a basis may not always exist.) Then the calculations of [5, Section 3.7] show that

$$\nabla_{Y_i} \frac{\partial}{\partial t} = \frac{\dot{f}_i}{f_i} Y_i. \tag{3.5}$$

Moreover, as $Y_i$ is Killing and $\xi_a$ is invariant with respect to the group action, we have on $\widehat{M}$

$$\nabla_{Y_i} \xi_a = \nabla_{\xi_a} Y_i, \tag{3.6}$$

and, on $I \times (eK)$, this vector field is equal to $(1/h_a)\nabla_{X_a} Y_i$ by (3.4).

Combining (3.2), (3.5) and (3.6) we obtain, on $I \times (eK)$, the equation

$$J_a Y_i = \frac{f_i}{h_a \dot{f}_i} \nabla_{X_a} Y_i$$

The covariant derivative on the left-hand side may be calculated using formula (3.16) of [5] (see also [6, Ch. 7]). Exploiting the orthonormality of $Y_i$ with respect to the Killing form, we obtain, on $I \times (eK)$,

$$J_a Y_i = \frac{f_i}{2h_a \dot{f}_i} \sum_j B_{ai}^j \frac{(f_j^2 - f_i^2 + h_a^2)}{f_j^2} Y_j, \tag{3.7}$$

where $B_{ai}^j$ are the structure constants defined by

$$\mathrm{ad}(X_a) Y_i = \sum_j B_{ai}^j Y_j.$$

## 4. Metrics when $G$ is not of type $A_n$

We can now study the possibilities for hyperkähler metrics on the manifolds discussed in Section 2. Theorem 2.1 shows that $\widehat{M}$ is an open set in $SL(n+1, \mathbb{C})/GL(n, \mathbb{C})$ or $\mathcal{U}(N)$ for some compact Wolf space $N$. If $N$ is a Wolf space we can write $N$ as $G/KSp(1)$. All the $G$-orbits in $\mathcal{U}(N)$ are now principal and are copies of $G/K$. There is exactly one compact Wolf space for each Lie algebra $\mathfrak{g}$ of compact simple type (except $\mathfrak{su}(2)$).

In this section we consider the case when $\widehat{M}$ is open in $\mathcal{U}(N)$, and $G$ is symplectic, orthogonal or one of the exceptional groups. The isotropy representation of $G/K$ is now the sum of a three-dimensional trivial summand isomorphic to $\mathfrak{su}(2)$ and a $4n$-dimensional irreducible summand $V$ [6]. The inner product on $V$ is therefore a scalar multiple $f^2$ (depending on $t$) of the Killing form, so a basis $Y_i$ of the sort discussed in Section 3 exists. Moreover all the $f_i^2$ are here equal to $f^2$.

Formula (3.7) for the complex structure now shows that

$$J_a Y_i = \frac{h_a}{2f\dot{f}}\,\mathrm{ad}(X_a)Y_i.$$

Imposing the condition that $J_a$ defines an almost complex structure now forces each function $h_a$ to equal $\pm 2f\dot{f}$. After sign changes and translation of $t$, Eqs. (3.3) show that we can take all the $h_a$ to be $-\frac{1}{2}t$. It follows that $f^2 = \pm\frac{1}{4}t^2 + c$ for a constant $c$. Evaluating the exterior derivative of the Kähler form on tangent vectors to the $G$-orbits actually shows that the Kähler condition forces $c$ to be zero.

The metric on $M$ is now determined. It is, again following [18], the standard hyperkähler metric on $\mathcal{U}(N)$. This can only be extended to a complete metric in the case when $N$ is $\mathbb{HP}(n)$ and $M$ is flat quaternionic space, so we do not obtain any complete irreducible metrics.

## 5. Metrics when $G$ is of type $A_n$

If $M$ is open in $SL(n+1,\mathbb{C})/GL(n,\mathbb{C})$ or in

$$\mathcal{U}(\mathrm{Gr}_2(\mathbb{C}^{n+1})) = \mathcal{U}(SU(n+1)/S(U(2)\times U(n-1)))$$

then the principal orbit is $SU(n+1)/U(n-1)$. The isotropy representation is the sum of a three-dimensional trivial summand $\mathfrak{p}_0$ isomorphic to $\mathfrak{su}(2)$, and two copies $\mathfrak{p}_1, \mathfrak{p}_2$ of the standard representation of $U(n-1)$ on $\mathbb{C}^{n-1}$. As $\mathfrak{u}(n-1)\oplus\mathfrak{p}_0$ forms a Lie subalgebra of $\mathfrak{su}(n+1)$, we can assume that the vector space $\mathfrak{p}_1\oplus\mathfrak{p}_2$ (though not its direct sum decomposition) is preserved by the adjoint action of $\mathfrak{p}_0$. Viewing $\mathfrak{p}_1\oplus\mathfrak{p}_2$ as $\mathbb{C}^{n-1}\otimes\mathbb{C}^2$, this $\mathfrak{p}_0$-action is just the standard $\mathfrak{su}(2)$ action on the $\mathbb{C}^2$ factor.

As usual, the metric on the three-dimensional summand can be diagonalised, and the coefficients $h_a^2$, $a = 1, 2, 3$, satisfy Eqs. (3.3).

We say that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are *Killing-orthogonal* if they are orthogonal with respect to the Killing form $\langle\cdot,\cdot\rangle$. For any $t_0$, we can choose $\mathfrak{p}_1$ and $\mathfrak{p}_2$ to be Killing-orthogonal and orthogonal with respect to the metric at $t = t_0$. For general $t$, these spaces will still be Killing-orthogonal but not of course metric-orthogonal.

The metric on $\mathfrak{p}_1\oplus\mathfrak{p}_2$ at a general value of $t$ will be given by

$$g(v, w) = \langle\phi v, w\rangle,$$

where $\phi$ is a $U(n-1)$-morphism of $\mathfrak{p}_1 \oplus \mathfrak{p}_2$, symmetric with respect to $\langle\cdot,\cdot\rangle$. The dependence of the metric on $t$ arises via the dependence of $\phi$ on $t$.

With a suitable choice of Killing-orthonormal bases over $\mathbb{R}$ for $\mathfrak{p}_1$ and $\mathfrak{p}_2$, we can arrange that $\phi$ has the form

$$\begin{pmatrix} \lambda^2 & 0 & \mu & -\nu \\ 0 & \lambda^2 & \nu & \mu \\ \mu & \nu & \tau^2 & 0 \\ -\nu & \mu & 0 & \tau^2 \end{pmatrix},$$

where $\lambda$, $\mu$, $\nu$ and $\tau$ are real scalar multiples of the $(n-1)\times(n-1)$ identity matrix. We shall also use $\lambda$, $\mu$, $\nu$ and $\tau$ to denote the eigenvalues of these matrices, that is, we shall regard $\lambda$, etc. as real-valued functions of $t$. Our choice of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ discussed above means that at $t=t_0$, $\mu$ and $\nu$ vanish.

Our first task is to show that we can in fact diagonalise the metric for all $t$, not just at $t=t_0$. The idea is to use Ricci-flatness to show that once $\mu$, $\nu$ vanish at $t_0$ they vanish for all $t$. We first recall a formula from [5, Section 3.17], which shows that at $t_0$,

$$\operatorname{Ric}\left(X, \frac{\partial}{\partial t}\right) = \sum_i g\left(T_{Z_i}\frac{\partial}{\partial t}, [Z_i, X]\right), \tag{5.1}$$

where $X$ is an arbitrary Killing field, the vectors $Z_i$ form an orthonormal basis of Killing fields at $t_0$, and $T$ is the O'Neill tensor. The formulae of [5] enable us to evaluate the term in $T$ and we can rewrite (5.1) as

$$\operatorname{Ric}\left(X, \frac{\partial}{\partial t}\right) = \sum_{i,j} g\left(\operatorname{ad}(X)Z_i, Z_j\right)\frac{\partial}{\partial t} g\left(Z_i, Z_j\right). \tag{5.2}$$

In our case, we can choose the $Z_i$ to consist of $h_a^{-1}X_a$, for $a=1,2,3$, together with $\lambda^{-1}Y_i$, $i=1,\dots,2n-2$, and $\tau^{-1}Y_i$, $i=2n-1,\dots,4n-4$, where $Y_1,\dots,Y_{2n-2}$ and $Y_{2n-1},\dots,Y_{4n-4}$ are bases for $\mathfrak{p}_1$ and $\mathfrak{p}_2$ respectively, orthonormal with respect to the Killing form.

Now, we can find $X$ and $\tilde{X} \in \mathfrak{p}_0$ such that $\operatorname{ad}(X)$ and $\operatorname{ad}(\tilde{X})$ have matrices

$$\operatorname{ad}(X) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \qquad \operatorname{ad}(\tilde{X}) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

with respect to the basis $Y_i$, $i=1,\dots,4n-4$, for $\mathfrak{p}_1 \oplus \mathfrak{p}_2$. So $\operatorname{ad}(X)$ and $\operatorname{ad}(\tilde{X})$ interchange $\mathfrak{p}_1$ and $\mathfrak{p}_2$.

Let us consider Eq. (5.2), with $X$ as given above. As $\mathfrak{p}_0$ is metric-orthogonal to $\mathfrak{p}_1 \oplus \mathfrak{p}_2$ for all $t$, the terms where $Z_i \in \mathfrak{p}_0$ and $Z_j \in \mathfrak{p}_1 \oplus \mathfrak{p}_2$ are zero. Since the metric on $\mathfrak{p}_0$ is diagonal for all $t$, it is easy to check that the terms with $Z_i, Z_j \in \mathfrak{p}_0$ are also zero. Finally, the fact that $\operatorname{ad}(X)$ interchanges $\mathfrak{p}_1$ and $\mathfrak{p}_2$, and the metric-orthogonality of these two spaces at $t_0$, implies that the terms with $Z_i, Z_j \in \mathfrak{p}_1$ or $Z_i, Z_j \in \mathfrak{p}_2$ vanish at $t_0$.

The remaining terms are those with $Z_i \in \mathfrak{p}_1$ and $Z_j \in \mathfrak{p}_2$ or vice versa. We obtain the equation

$$(\tau^2 - \lambda^2)\frac{\partial}{\partial t}\left(\frac{\mu}{\lambda\tau}\right) = 0 \quad \text{at } t = t_0. \tag{5.3}$$

If the metric on $\mathfrak{p}_1 \oplus \mathfrak{p}_2$ is proportional to the Killing form for all $t$, we may take any Killing-orthonormal basis for $\mathfrak{p}_1 \oplus \mathfrak{p}_2$ and diagonalise the metric with respect to a basis of Killing fields.

In all other cases we can choose $t_0$ so that, in our notation, $\lambda^2(t_0) \neq \tau^2(t_0)$. As $\mu$ vanishes at $t_0$, we see from (5.3) that its first derivative does also. Using $\tilde{X}$ in Eq. (5.2) we get a similar conclusion for $\nu$. We have shown that once the metric is diagonalised at $t_0$, its first derivative at $t_0$ is also diagonal.

The formulae of [5, Sections 3.6, 3.7, 3.9 and 3.11] enable us to express the Ricci tensor at $t_0$ evaluated on Killing fields in terms of the metric and its first and second derivatives. The second-order ODE thus given by the vanishing of the Ricci tensor now implies that the off-diagonal terms of the metric in fact vanish at $t_0$ to all orders.

The metric is therefore diagonalised for all $t$ by a basis of Killing fields, and we can now apply the arguments of Section 3.

Eq. (3.7) shows that, for $a = 1, 2, 3$, the endomorphism $J_a$ is given on $\mathfrak{p}_1 \oplus \mathfrak{p}_2$ by

$$J_a = \begin{pmatrix} \alpha_a P_a & \beta_a Q_a \\ \gamma_a R_a & \delta_a S_a \end{pmatrix},$$

where

$$\mathrm{ad}(X_a) = \begin{pmatrix} P_a & Q_a \\ R_a & S_a \end{pmatrix}.$$

Here $\alpha_a, \dots, \delta_a$ are scalar functions, given by

$$\alpha_a = -h_a/2\lambda\dot{\lambda}, \tag{5.4}$$

$$\beta_a = \frac{\tau(\tau^2 - \lambda^2 - h_a^2)}{2h_a\lambda^2\dot{\tau}}, \tag{5.5}$$

$$\gamma_a = \frac{\lambda(\lambda^2 - \tau^2 - h_a^2)}{2h_a\tau^2\dot{\lambda}}, \tag{5.6}$$

$$\delta_a = -h_a/2\tau\dot{\tau}. \tag{5.7}$$

If we denote by $\Omega_a$ the Kähler form associated to $J_a$, the Kähler condition implies that

$$\frac{\partial}{\partial t}\Omega_a(Y_i, Y_j) + \Omega_a\left(\frac{\partial}{\partial t}, [Y_i, Y_j]\right) = 0 \tag{5.8}$$

(the left-hand side of this equation is just $\mathrm{d}\Omega_a(\partial/\partial t, Y_i, Y_j)$). If $i, j \leq 2n-2$, (5.8) together with the above expressions for $J_a$ implies that

$$\dot{\alpha}_a\lambda^2 B_{1i}^j = 0. \tag{5.9}$$

Now, our discussion earlier of the $\mathfrak{su}(2)$-action on $\mathfrak{p}_1 \oplus \mathfrak{p}_2$ showed that $\mathrm{ad}(X_1)$, $\mathrm{ad}(X_2)$ and $\mathrm{ad}(X_3)$ cannot all interchange $\mathfrak{p}_1$ and $\mathfrak{p}_2$. Without loss of generality, we may assume that $\mathrm{ad}(X_1)$ does not interchange these spaces. It follows that $P_1$ is non-zero, and there exist $i, j \leq 2n-2$ such that $B^j_{1i} \neq 0$. We deduce from (5.9) that $\alpha_1$ is constant, and a similar argument shows that $\delta_1$ is constant.

We now claim that $\mathrm{ad}(X_2)$ or $\mathrm{ad}(X_3)$ interchanges $\mathfrak{p}_1$ and $\mathfrak{p}_2$. For if not, the same argument as in the preceding paragraph shows that $\alpha_2$ and $\alpha_3$ are constant. Eq. (5.4) now implies that the ratios $h_a/h_b$ are all constant, and it easily follows from (3.3) that $h_1^2 = h_2^2 = h_3^2$. In this situation the diagonal form of the metric on $\mathfrak{su}(2)$, and the relations $[X_1, X_2] = X_3$, etc. are preserved by orthogonal changes of basis of $\mathfrak{su}(2)$. A suitable such basis change will ensure that $\mathrm{ad}(X_2)$ interchanges $\mathfrak{p}_1$ and $\mathfrak{p}_2$.

Our claim is now proved. Without loss of generality we can in fact choose $\mathrm{ad}(X_2)$ to be the element interchanging the two summands.

Next, we claim that we can take $\mathrm{ad}(X_3)$ to interchange $\mathfrak{p}_1$ and $\mathfrak{p}_2$. The argument is similar to that above. If $\mathrm{ad}(X_3)$ does not interchange $\mathfrak{p}_1$ and $\mathfrak{p}_2$, then in the usual way we deduce that $\alpha_3$ is constant. From above we know $\alpha_1$ is constant, so from (5.4) we deduce that $h_1/h_3$ is constant. Eqs. (3.3) now show that $h_1^2 = h_3^2$, so orthogonal transformations of $\{X_1, X_3\}$ preserve the diagonal nature of the metric and the $\mathfrak{su}(2)$ commutation relations. After such a transformation, we can assume that $\mathrm{ad}(X_3)$ interchanges $\mathfrak{p}_1$ and $\mathfrak{p}_2$.

To sum up, we can assume that $\mathrm{ad}(X_2)$ and $\mathrm{ad}(X_3)$ interchange the spaces $\mathfrak{p}_1$ and $\mathfrak{p}_2$. The relation $[X_2, X_3] = X_1$ now shows that $\mathrm{ad}(X_1)$ preserves $\mathfrak{p}_1$ and $\mathfrak{p}_2$. As $J_1$ is Hermitian we deduce that $\alpha_1 = \pm 1$. Similarly, as $J_2$, $J_3$ are Hermitian, we have $\beta_2 = \pm\tau/\lambda$, $\beta_3 = \pm\tau/\lambda$ and $\gamma_2 = \pm\lambda/\tau$, $\gamma_3 = \pm\lambda/\tau$.

The Kähler condition (5.8) for $J_2$ implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma_2\tau^2) + h_2 = 0, \tag{5.10}$$

so we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\pm\lambda\tau) + h_2 = 0.$$

Similarly, considering $J_3$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\pm\lambda\tau) + h_3 = 0$$

and it follows that $h_2^2 = h_3^2$.

Using (3.3) we find that the functions $h_1, h_2, h_3$ are now expressible in terms of trigonometric or hyperbolic functions. Defining a new variable $s$ by $\mathrm{d}s/\mathrm{d}t = (h_1h_2h_3)^{-1}$, we have one of the following three possibilities:

$$h_1 = -D\cot(Ds), \qquad h_2 = h_3 = -D\,\mathrm{cosec}\,(Ds), \tag{5.11}$$

$$h_1 = -D\coth(Ds), \qquad h_2 = h_3 = -D\,\mathrm{cosech}\,(Ds), \tag{5.12}$$

or

$$h_1 = h_2 = h_3 = -1/s, \tag{5.13}$$

where $D$ is a real constant. We can view (5.13) as the $D \to 0$ limit of (5.12).

After appropriate sign changes, we can without loss of generality take $\alpha_a = 1$, $\gamma_a = \lambda/\tau$ etc. Our equations then imply

$$\frac{\mathrm{d}(\lambda^2)}{\mathrm{d}t} = -h_1, \quad \frac{\mathrm{d}(\tau^2)}{\mathrm{d}t} = -h_1, \quad \frac{\mathrm{d}(\lambda\tau)}{\mathrm{d}t} = -h_2,$$

and from (3.3) we can rewrite these as

$$\lambda^2 = h_2h_3 + K_1, \quad \tau^2 = h_2h_3 + K_2, \quad \lambda\tau = h_1h_3 + K_3,$$

for some constants $K_1$, $K_2$, $K_3$.

Substituting in our expressions for $h_a$, we find that $K_3 = 0$, $K_1 = -K_2$ and the $h_a$ are given either by (5.13) or (5.12) with $D^2 = K_1^2$.

The upshot is that we have a family of metrics parametrised by a non-negative number $D$ (changing the sign of $D$ does not change the metric). If $D$ is non-zero, these are the Calabi metrics or their restrictions. If $D = 0$, we get the metric of [18] on $\mathcal{U}(\mathrm{Gr}_2(\mathbb{C}^n))$, which does not extend to a complete metric.

## Acknowledgements

The first author thanks the I.H.E.S. for support in the early stages of this work and Aarhus University for its hospitality during the Geometry and Physics conference in July 1995.

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